

CHIRALITY ON THE LATTICE *

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During the last several years a non-perturbative formulation of exact chiral symmetry on the lattice has been developed. I shall outline the main ideas of these developments and discuss prospects for the future. The focus will be on the basic concepts enciphered in a new jargon consisting of terms like “infinite number of fermions”, “domain wall fermions”, “the overlap”, “the Ginsparg-Wilson relation”. Technical details will be omitted.

1 Introduction

As far as we know the basic constituents of matter are chiral fermions. Their interactions are described by an effective theory, the minimal standard model. This model is experimentally established at relatively long scales. The scales where field theory will cease being an adequate description of Nature are much shorter. Thus, there is a large range of scales where field theory is reliable. A complete theory would contain all scales. We are led¹ to ask: Is it plausible for a complete theory to contain such a large separation of scales, given that the long scales are chiral? If we restrict the question to renormalized, perturbative field theory, the answer is positive. But, if we now consider the same question in a non-perturbative context, the answer is less clear.

To simply ignore the non-perturbative aspects of the question would be wrong: Non-perturbative considerations have proven to be relevant before, for example providing an intrinsic upper bound on the Higgs mass (the triviality bound)². Chiral gauge theories pose difficulties to non-perturbative analysis. The most famous of these is encountered in lattice field theory. For a long time it was believed that chiral gauge theories cannot be defined in any reasonable way on the lattice³. Developments over the last six years are poised to falsify this belief.

The main physics question is then what can be learned from the difficulties on the lattice and their resolution: Are the problems general or just a specific lattice quirk? Does the recent resolution point to a generic structure, relevant also off the lattice, in the real world? Most experts would agree that the problem is not just a lattice quirk; nevertheless, most field theorists work in the continuum, and implicitly assume otherwise. Accepting that the problem is generic we would expect a resolution that is not limited to the lattice. The recent resolution indeed is not.

In a nutshell, it works by postulating an infinite number of extra heavy fermions per unit four-volume. All the extra fermions are as heavy as we wish. Their masses can

be kept naturally at energies many orders of magnitude above the typical energy-momentum scales of the chiral gauge theory governing the long scales. They can be integrated out fully, leaving a consistent low energy chiral theory. The required infinite number of fermions is suggestive of one or more extra dimensions, beyond the four we know. It is tantalizing that there exist other speculations about the fundamental laws of Nature that also use extra dimensions.

This solution was developed in collaboration with Narayanan^{4,5,6}. We started to work on the problem in response to independently conceived ideas that appeared in two papers. The most widely known paper⁷ is by Kaplan and has roots in earlier work⁸ by Callan and Harvey. The other paper⁹ was authored by Frolov and Slavnov. These two papers seemed very different but Narayanan and I identified a common mechanism. We pursued these insights to a concrete realization. Recently it has become clear that some germs of the idea can be found in an earlier paper by Ginsparg and Wilson¹⁰.

During the last year many more lattice field theorists have started working on implementations of exact chiral symmetry on the lattice. Progress has been particularly rapid for vector-like field theories, where the exact chiral symmetries are global. This is important for QCD. Although in QCD the quarks are massive, there are chiral symmetries that are approximate and their incorporation in effective descriptions of the low energy properties of QCD has been extremely successful phenomenologically. Thus, numerical work on lattice QCD would benefit enormously from a practical lattice formulation of exact global chiral symmetry.

2 Different views of infinite numbers of fermions

Callan and Harvey were looking for a physical setup to find connections between anomalies in dimensions that differ by one or two units. They considered a gauge theory in five dimensions with a mass term that depended on the fifth dimension going monotonically from a positive value to a negative one. The five dimensional free Dirac equation has solutions that propagate with the speed of

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light along the wall but are exponentially suppressed in directions perpendicular to the wall. These solutions also turn out to be unpaired eigenstates of the γ_5 -matrix. They represent chiral fermions confined to the four dimensional domain wall located at the point x_5 where the mass vanishes. The effective action of the fermions on the wall is chiral, so can be anomalous. The associated charge must leak into the fifth dimension since from the five dimensional point of view the charge is conserved. This charge has to be allowed to disappear at the \pm infinities in the fifth direction; one cannot compactify that dimension, as charge would then have no way of disappearing. The number of fermions is infinite not only because the fifth direction is continuous, but, more importantly, because it is open at the two ends.

Kaplan showed that the Callan-Harvey phenomenon could be realized on a lattice using a Wilson mass term to create the domain-wall, the “defect”. He understood that the continuity of the fifth direction (in addition to that of the first four) was not essential. But, he abandoned the openness of the ends of the fifth direction in order to make the whole setup finite. This added an anti-domain-wall housing fermions of opposite chirality. Kaplan speculated that one could devise a gauge action which could keep the chiral fermions at their respective walls and the two chiralities would not communicate even in the presence of arbitrary gauge fields. So, one still had a vector-like theory but the two chiral components of the fermions were “physically separated” in the fifth direction and, if it so happened that there were no anomalies at one of the walls, one could hope that the walls would dynamically decouple. It still could happen that fermion number would be violated at one of the walls. Such violations had to be exactly compensated at the other wall - the decoupling of the walls was not complete. Before Kaplan, Boyanowski et al. ¹¹ discussed Callan-Harvey phenomena on the lattice, but their suggestions were less concrete and attracted no attention.

Frolov and Slavnov considered the regularization of $SO(10)$ gauge theory with one 16-plet of Weyl fermions. They introduced an infinite number of heavy fermion fields of normal and abnormal statistics. Their scheme made use of specific $SO(10)$ properties. The choice of statistics of the fermi fields ensured that the real part of the effective action induced by the fermions came in with the required weight, half of that induced by Dirac fermions. The ordering implied by the masses implied that the infinity was one dimensional. On the other hand, the gauge fields were just four dimensional. The entire treatment was in the continuum.

The papers by Kaplan and Frolov and Slavnov appeared at a time that the importance of topology induced fermion number violating processes was recognized. In a bilinear action different multiplets of fermions do not cou-

ple to each other. Therefore, one can imagine integrating over the multiplets independently. But, in an instanton background, something very strange must be allowed to happen: a single ψ field may acquire a nonzero expectation value. This is impossible if the kernel of the bilinear form is a finite square matrix. In the continuum, this is possible because of a non-zero index. The index of a linear operator can be understood as a measure of the difference between the number of rows and the number of columns of a finite matrix approximating the operator. This difference cannot be frozen: it must change as the gauge background changes. This can be achieved only if the kernel is an infinite matrix, in other words, the number of fermions is infinite.

Narayanan and I interpreted the papers of Kaplan and Frolov and Slavnov in a new way. Our setup also accommodated the above described instanton phenomena. We start with ⁴ an apparently vectorial gauge theory:

$$\mathcal{L}_\psi = \bar{\psi} \gamma_\mu D_\mu \psi + \bar{\psi} \left(\frac{1 + \gamma_5}{2} \mathcal{M} + \frac{1 - \gamma_5}{2} \mathcal{M}^\dagger \right) \psi \quad (1)$$

The operator \mathcal{M} acts in a new space and has an analytic index there (do not confuse this index with the one associated with instantons): while $\dim(\text{Ker } \mathcal{M})=1$, $\dim(\text{Ker } \mathcal{M}^\dagger)=0$. This property is radiatively stable, but can be realized only for $\dim \mathcal{M}=\infty$. The infinite size of \mathcal{M} and its index make it impossible to use a standard bi-unitary transformation on the fermi fields to replace a non-hermitian \mathcal{M} by a hermitian operator. If in Kaplan’s original approach one made the gauge fields four dimensional his scheme looked just as Frolov and Slavnov’s only the mass matrices were different. In his case the masses approached asymptotic values, while Frolov and Slavnov chose them to increase without bound. But, in both cases one had an index.

To control the infinity we interpreted the fifth direction as an imaginary time coordinate. So, we had to calculate the projectors on the vacua of the many-fermion problems associated with asymptotic translations in the fifth direction. Starting from the \pm infinities, two different states propagate inwards, towards the defect. The fermion-induced action of interest is localized at the interface and is contained in the overlap of the two states - hence the term “overlap” ⁵.

The projectors are onto rays: thus, while the absolute value of the overlap is well defined its phase is not. This is just right: gauge invariance is preserved at all steps and because of anomalies it has to be impossible to achieve a complete definition for arbitrary fermion representations. We have a $U(1)$ bundle over the space of gauge fields and the bundle itself can be reduced to the space of gauge orbits. But, there is no guarantee that reasonable sections of the bundle can be found: We ended

up with a mathematical structure able to reproduce the subtle features of chiral fermions⁶.

Ginsparg and Wilson proposed in 1982 a renormalization group approach to representing exact global chiral symmetries on the lattice. The renormalization group is designed to bring out quantum versions of scale invariance which are anomalous. Massless QCD is classically scale invariant and consequently has exact global chiral symmetry. Therefore, a fixed point for massless QCD should exhibit chiral symmetry and also chiral anomalies. For concreteness, Ginsparg and Wilson imagined starting the renormalization group iteration from a chirally symmetric action. The iteration has to break chiral symmetry to make anomalies possible, but this is the single source of breaking. The initial symmetry constrains the resulting effective action. This action has to satisfy a remnant of chiral symmetry, the Ginsparg-Wilson relation. From it, Ginsparg and Wilson derive the anomalous $U(1)_A$ Ward identity. In footnote 11 they observe that all non-anomalous Ward identities could be derived in the same manner. Such identities hold when more than one quark flavor is present ($N_f > 1$). In footnote 7 they comment that the fixed point action cannot be bilinear in the fermion fields when $N_f > 1$ ¹².

Their paper was forgotten because, in the presence of gauge fields, they had no explicit solution to their relation. That solutions should exist for $N_f = 1$ was plausible if one accepted the existence of renormalization group transformations with acceptable bilinear fixed points. To reach a fixed point an infinite number of iterations are required. Each iteration removes a slice of short ranged fermion modes and compensates by dilating the remainder. The chirally symmetric starting point is then separated from the actual action by the elimination of an infinite number of fermionic (and bosonic) degrees of freedom.

I have mentioned already that the infinite matrix \mathcal{M} can realize both Kaplan's and Frolov and Slavnov's ideas and in its Kaplan-version leads to the overlap. In the vector-like context, if one does not insist on explicit decoupling of left and right Weyl spinors in the action, the kernel of the bilinear fermion can have an unrestricted square shape for any gauge background. Thus, one can hope for a sequence of kernels of finite square shape that converge to the infinite kernel representing the strictly massless case. Such a sequence was known since Kaplan: it is nothing but his domain wall construction, slightly modified^{11,13}. Viewed differently, it consists of a light Dirac fermion coupled to many heavy Dirac fermions¹⁴ by a mass matrix of seesaw type. The seesaw suppression produces strictly massless fermions only after an infinite amplification. But, the mass decreases exponentially fast.

I have emphasized the infinite number of fermions

because it is a physically appealing picture. Mathematically however, the overlap construction⁵ was recast⁶ in a way that made no references to anything infinite. As a result, some technical simplifications were achieved. Moreover, the scheme is very flexible. This flexibility amounts to freedom to choose from large classes of lattice H -operators. Adapting the form of H to the specific problem at hand turned out very useful for efficient numerical simulations in two dimensions and also in analytical work. This flexibility will very likely be further exploited in future applications.

3 Finite number of fermions

In the vector-like case the overlap admits an explicit form involving a finite square matrix, the overlap-Dirac operator¹⁵, which couples left and right Weyl components, and is therefore of the type that should obey the Ginsparg Wilson relation. It should do so because it was obtained by infinite iteration from an explicitly chiral starting point. It indeed does^{14,16}, and thus we finally have an explicit solution to this relation and the properties established by Ginsparg and Wilson hold.

More in line with the original work it has been recently claimed that a true fixed point to full massless QCD can be replaced by a classical approximation, called "a perfect action"¹⁷, and the Ginsparg Wilson relation still holds. The perfect action is bilinear in the fermions for any number of flavors, but no explicit expression is known. An explicit definition is provided only for the "perfect" renormalization group transformation and any fixed point of this map is a "perfect action". The map contains a minimization step which introduces a non-analyticity in the background. Some singularities are necessary, because of instantons¹⁶. In the overlap-Dirac operator the singularities are in direct correspondence with exact zero eigenvalues of a finite, local and analytic matrix.

At present it is unclear whether another class of explicit solutions to the Ginsparg Wilson relation will be found. There are obvious variations on the overlap itself, but it seems hard to find something explicit and really new. There also are indications^{18,19} that all acceptable solutions to the Ginsparg Wilson relation have an overlap "flavor".

The infinite number of fermions imply a dependence only on the rays making up the overlap. On a finite lattice these rays are points in a $CP(N)$ or $RP(N)$ space and their Berry phases are the mathematical vehicle bringing in unavoidable anomalies and the need for their cancellation²⁰. Anomalies can show up because the infinite number of fermions introduces a lack of determinacy. Without it, the regulated chiral determinant would be a function of gauge fields, rather than a quantity de-

fined up to phase. The phase freedom is restricted by requiring the states in the overlap to depend smoothly on the gauge field. This requirement cannot be made compatible with gauge invariance if anomalies do not cancel. On the other hand, the real part of the induced action is gauge invariant, no matter what happens with anomalies.

If one wishes to work within a scheme that makes no reference to infinite numbers of fermions, the needed rays can be introduced by hand. However, solutions to the Ginsparg Wilson relation, unlike the overlap Hamiltonians or true fixed point actions, cannot be smoothly dependent on the gauge fields. Therefore, even if one extracts the relevant subspaces, it becomes unclear why one should care about Berry phases, given that the associated operators depend non-analytically on the gauge fields. To consider Berry phases we should allow ourselves to be aware of the smooth overlap-Hamiltonians. What distinguishes overlap solutions to the Ginsparg Wilson relation is that these come from analytic Hamiltonians and consequently have only singularities of a certain type. It seems contrived to eliminate these Hamiltonians but keep the singularity structure. In any case, an approach ostensibly based solely on the Ginsparg Wilson relation, but exploiting spectral representations of the fermion kernel ends up being equivalent to the overlap construction.

Ginsparg and Wilson knew nothing about the overlap and simply assumed the existence of an explicitly chirally symmetric starting point which had to be left somewhat ill defined. It is clear now that one can proceed this way and get a well defined scheme in the end. But, should we really ignore the starting point of the iteration and just focus on the relation observed by the fixed point? I think not: Nature has no reason to first come up with a Ginsparg Wilson relation and then find a solution - this relation is either obeyed or not, but the reason must be elsewhere, at a deeper level.

4 Main achievements of the overlap

The overlap has been extensively tested both on and off the lattice.

Explicit computations^{6,21} in perturbative gauge backgrounds confirmed that the fermions were indeed chiral. In perturbation theory the generality of the overlap structure was made evident by calculations in non-lattice regularization schemes. These continuum schemes also simplified the needed algebra. Perturbation theory was used to compute anomalies, the vacuum polarization, and to check²² the radiative stability of masslessness. It produced both consistent and covariant anomalies, pinpointed the source for their difference, and provided insights that could later be abstracted outside perturbation theory. Also, even in the vector-like context²³ it was necessary to see how various “no-go” theorems were avoided

at the level of perturbation theory.

Numerical work established that instanton effects were correctly reproduced^{6,24,25}. Rather vexing questions had to be answered. Was it indeed true that a single fermion could acquire an expectation value in an instanton background? How could one have explicit violation of $U(1)_A$ in the vector-like context without violating any other global axial symmetry? Eventually it became clear that a fully regulated version was available where 't Hooft's solution to the QCD $U(1)_A$ problem was valid.

For chiral models instanton effects are more dramatic. In two dimensions it was shown that fermion number violation is reproduced in the overlap²⁶. The model that was investigated also has composite massless fermions and provides a simple example where 't Hooft's consistency conditions are non-trivially respected. Using the overlap, this model was simulated numerically, and the success of this experiment constitutes the most subtle test of the overlap to date. The test also shows that exact gauge invariance on the lattice is not needed so long as the model makes sense in the continuum²⁷. A slight breaking of gauge invariance amounting to short ranged correlations between the gauge degrees of freedom $g(x)$ is irrelevant in the continuum limit where the $g(x)$ become independent and decouple from observables.

Non-perturbative anomalies are relatively subtle in continuum physics. In a non-perturbative setting they should become simpler to understand than perturbative anomalies. This was shown for the overlap in three and four dimensions^{28,29}. In both dimensions the mechanism behind the anomaly is simple: the non-perturbative anomalies reflect Berry phase obstructions to choosing global phases of the states making up the overlap. These obstructions are directly related to overlap-Hamiltonian level crossings over an extended parameter space.

Three dimensions is a promising area of applications for the overlap^{28,30}. Although there is no chirality in any odd dimensions, there exists an analogue, and the overlap machinery can be extended to three dimensions with ease. A particular set of three dimensional models that were studied by Appelquist³¹ and collaborators, admit an overlap formulation. From it, I derived a three dimensional generalization of the Ginsparg Wilson relation. Three dimensional models will be instructive.

5 Prospects

The initial apathetic reaction of dominant factions in the lattice community to the progress on the chiral fermion front has all but disappeared. The most convincing sign of change are the emerging priority squabbles.

If the overlap is correct and practical difficulties are overcome, the way lattice QCD is currently being done

will undergo a revolution. Currently, a large fraction of the numerical QCD effort, which commands the bulk of support, manpower and visibility in lattice field theory, is invested in diminishing and controlling chirality violating effects. Following initial work on a two dimensional vector-like model³², serious studies³³ of the domain wall truncation of the overlap have been undertaken and consequently the largest computer resources in the US will be applied to the domain-wall-fermion seesaw approximation of the overlap³⁴.

Even more recently, progress has been made also on the direct implementation of the overlap-Dirac operator on the lattice^{30,35}, relegating all truncations to their natural place: numerical algorithms. It is too early to say whether the direct approach or the domain wall one will ultimately prove more efficient. In principle, the direct approach is cleaner.

6 Summary

A new and exciting lattice methodology is emerging and, possibly, we are witnessing a big step forward in numerical QCD. At the base of this progress lies a world consisting of an infinite number of fermions, all but one having very large masses. This infinity is fully under control and consequently can be completely eliminated. It is natural to speculate that we have discovered more than just a trick designed for computers, namely, that we have obtained a valuable hint about chirality in Nature.

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